Lecture 11

Limits

We say that L is the limit of f at (x_0, y_0, z_0) if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t.

if $0 \leq \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta \rightarrow |f(x,y,z) - L| < \varepsilon$

We write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

and say the limit exists.

Continuity

The function f is continuous at (x_0, y_0, z_0) if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0)$$

Partial Derivative

The partial derivative of f w.r.t. x @ (x_0, y_0) is define as:

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}.$$

Similarly, the partial derivative w.r.t. y @ (x_0, y_0) is:

$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

We also talk about partial derivative functions.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

We also frequently use the notation:

$$\frac{\partial f}{\partial x} = f_x(x,y) \quad \& \quad \frac{\partial f}{\partial y} = f_y(x,y)$$

Ex. 1

If $f(x,y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2,1)$ and $f_y(2,1)$.

$f_x = 3x^2 + 2xy^3$ $\qquad$ $f_y = 3x^2y^2 - 4y$

$f_x(2,1) = 16$ $\qquad$ $f_y(2,1) = 8$

Partial Derivative Rules

$(f+g)_x = f_x + g_x$

$(f-g)_x = f_x + g_x$

$(fg)_x = f_x g + f g_x$

$$\left(\frac{f}{g}\right)_x = \frac{f_x g - f g_x}{g^2}$$

Ex. 2

With $f(x,y) = \dfrac{x^3y - xy^3}{x^2+y^2}$, find f_x & f_y

$$f_x = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$$

$$f_x = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_y = \frac{(x^3 - 3xy^2)(x^2+y^2)(x^3y - xy^3)(2y)}{(x^2+y^2)^2}$$

$$f_y = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

Ex. 3

Let $f(x,y,z) = e^{2x}\cos(z) + e^{3y}\sin(z)$. Find the partial derivatives.

$f_x = 2e^{2x}\cos(z)$

$f_y = 3e^{3y}\sin(z)$

$f_z = -e^{2x}\sin(z) + e^{3y}\cos(z)$

Higher Order Partials

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2}{\partial x^2} f = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$f_{xy} = \frac{\partial^2}{\partial y \partial x} f = (f_x)_y$$

$$f_{yx} = \frac{\partial^2}{\partial x \partial y} f = (f_y)_x$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

Ex. 4 Find the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$

$f_{xx} = 6x + xy^3$

$f_{xy} = 6xy^2$

$f_y = 3x^2y^2 - 4y$

$f_{yy} = 6x^2y$

$f_{yx} = 6xy^2$

Clairaut's Theorem

If f_{xy} & f_{yx} are both continuous on a disk D containing (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$